## Formal Logic III

Background

This is a continuation of Formal Logic II. The aims are:

propositional logic – declarative sentences, natural deduction, semantics, normal forms

predicate logic – syntax, natural deduction, semantics, undecidability

modal logics – syntax, semantics, logic engineering, natural deduction, multi-agent systems

**Lesson 0**

Propositional logic  
 : Susan saves

: Susan buys a house

: The house has seven bathrooms

: The house is used as a guesthouse

: Peter likes the house

**“only if”**

“The house is used as a guesthouse sonly if it has seven bathrooms”

“Susan buys a house only if the house does not have seven bathrooms”

**“unless”**

“if… then”

“Unless Susan saves, she does not buy a house with seven bathrooms”

“Unless Peter does not like the house, Susan saves and buys a house used as guesthouse”

*Negate one side of the statement*

**“if and only if”**

“logically equivalent to”

“The house is used as a guesthouse if and only if Susan does not buy it, but Peter likes it”

“Peter likes the house if and only it has seven bathrooms or is not used as guesthouse”

*Remember “but” in logic is treated as “AND”*

**Lesson 0**

Symbols

|  |  |  |
| --- | --- | --- |
| **Symbol** | **Name** | **Description** |
|  | phi | Premise/formula |
|  | psi | Conclusion |
|  | Proves | Proves/implies/yields. (Syntactically entails) |
|  | True | Satisfies (models) |
|  | Sequent | Is valid if a proof can be found |
|  |  |  |

We apply proof rules to premises, we get formulas.

We apply more proof rules to formulas, we get a conclusion

Hence:

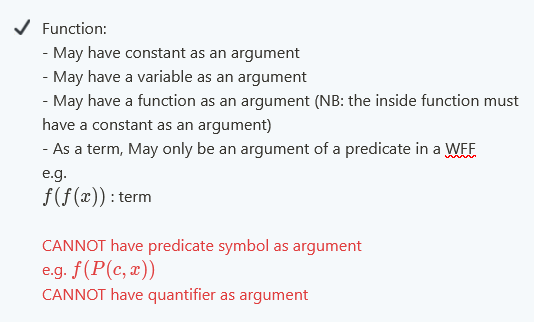
Example:

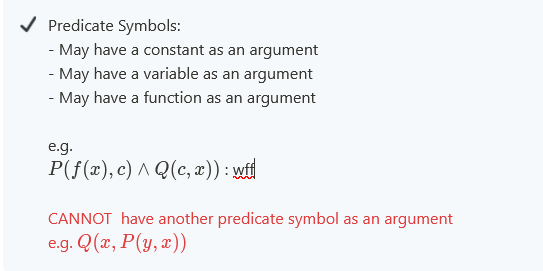
Satisfiability:

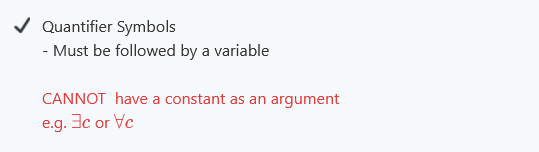
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| --- | --- | --- |
| **Symbol** | **Name** | **Description** |
|  | Phi | Premise/formula |
|  | Psi | Conclusion |
|  | True | Satisfies (models) |
|  | Model M |  |
|  | Environment |  |
|  | Satisfaction relation |  |
|  | Semantic entailment | Is valid if a proof can be found |

**Lesson 0**

WFF : Well-Formed Formulae







**Lesson 1**

Natural Deduction Rules

Natural deduction is used to examine the validity of arguments. It can sometimes be more efficient than a truth table. This is like Fitch Rules, where the goal is to build a proof, using some type of ruleset.

Recap:

Elim: Eliminate or remove parts

Intro: Introduce or add parts

Example

*Prove that implies*

1 assume a & b.

2 b. &E2 1

3 a. &E1 1

4 b & a. &I 2,3

5 therefore a & b => b & a. =>I 1,4

Conjunction ()

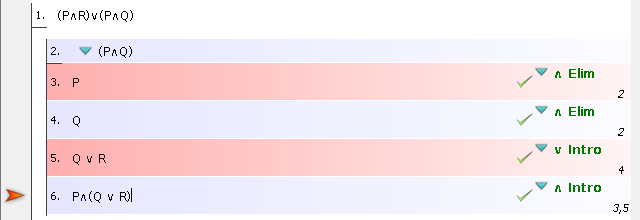
: I am happy

: The sun is shining

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Sentences to cite | Usage |
|  | **And\_Intro.**  Replaces the current goal by the two goals A and B  *I am happy, and the sun is shining*  *is true. is true. Implies is true* | Any. May cite as many prior lines as you like, and each will be a conjunct. | Introduce a new conjunction on any line of a proof by citing each of the conjuncts from prior lines.  These conjuncts must be alone on the line cited. |
|  | **And\_Elim\_1 in H.**  Applies to an assumption of the form H:  And generates a new assumption HO:  *I am happy*  *is true. Implies is true* | (*Elim): for each conjunct, the assumption.*  *Enter every conjunct from the assumption. On a new line* | Remove a conjunct from a previous line containing a conjunction. |
|  | **And\_Elim\_2 in H.**  Applies to an assumption of the form H:  And generates a new assumption HO:  *The sun is shining*  *is true. Implies is true* | (*Elim): for each conjunct, the assumption.*  *Enter every conjunct from the assumption. On a new line* | Remove a conjunct from a previous line containing a conjunction. |
|  | **And\_Elim\_all in H.**  Applies to an assumption of the form H:  And generates two assumptions  H:  HO:  The tactic is then recursively applied to H and HO |  |  |

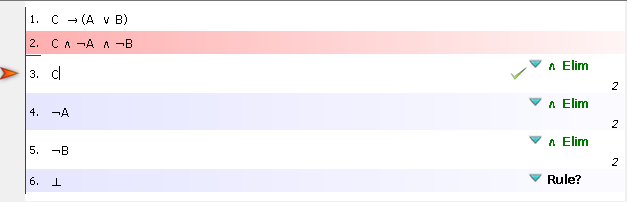
*Intro: Cite as many sentences as you like*

*All must be conjuncts*



*Elim: for each conjunct, cite 2.*

*Enter every conjunct from 2. On a new line*



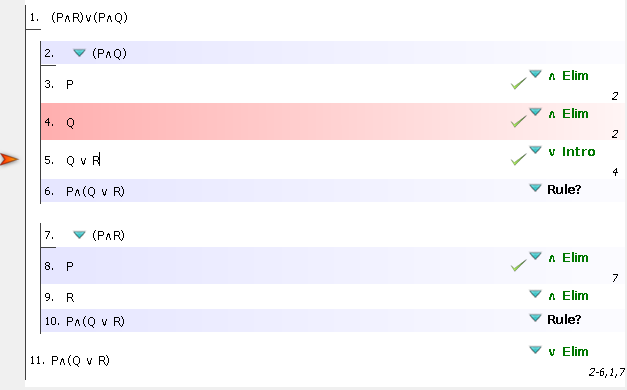
Disjunction ()

: I am happy

: The sun is shining

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Sentences to cite | Usage |
|  | **Or\_Intro\_1**  Replaces the current goal by the goal | Any. Cite only one prior line, it will be a disjunct. | You can cite any prior sentence available and create a disjunction containing as one conjunct the prior line cited and as another disjunct any sentence you like. |
|  | **Or\_Intro\_2.**  Replaces the current goal by the goal | Any. Cite only one prior line, it will be a disjunct. | You can cite any prior sentence available and create a disjunction containing as one conjunct the prior line cited and as another disjunct any sentence you like. |
|  | **Or\_Elim in H.**  Applies to an assumption of the form H:  It generates two proof obligations with the assumptions  H: resp.  H: and the current goal | *(∨ Elim) Cite Assumption. And each subproof* | *Each disjunct should be subproof*  *Each subproof should have the same goal as the other*  *The goals in the subproofs should match a new goal outside* |

*(∨ Intro). Cite only one prior line, it will be a disjunct.*

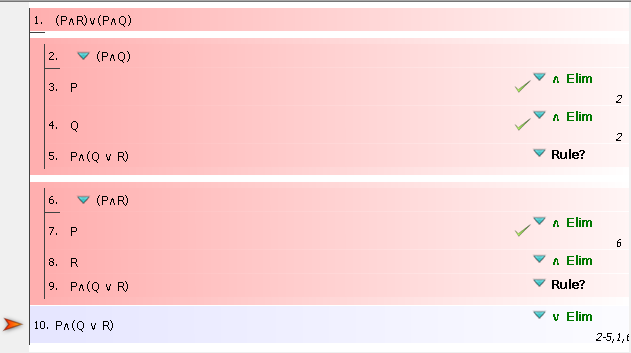


*(∨ Elim) Cite 1. And each subproof*

*Each disjunct should be subproof*

*Each subproof should have the same goal as the other (in this case,)*

*The goals in the subproofs should match a new goal outside (in this case, our intermediate conclusion ,))*

****

Implication ()

: I am happy

: The sun is shining

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Sentences to cite | Usage |
|  | **Impl\_Intro.**  Replaces the current goal by and adds the assumption H: | You must cite only a single subproof. | To prove a conditional statement, make a subproof that begins with the antecedent and ends with the consequent. |
|  | **Impl\_Elim in H and HO.**  Applies to the two assumptions of the form H: and HO: and adds the new assumption H1: | You must cite exactly two sentences:   1. a conditional and 2. a sentence that is the antecedent of the conditional in 1. | You can only prove the consequent of the conditional cited in 1. above. |

*-> Intro:*

*Add a new subproof*

*The assumption of the subproof should be the antecedent ()*

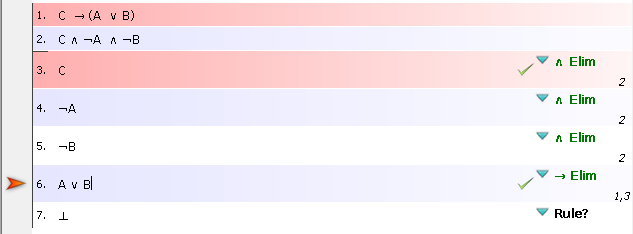
*The conclusion of the subproof should be the consequent ()*

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*-> Elim: Cite a conditional (1.)*

*and a sentence that’s an antecedent of the conditional (3.)*

****

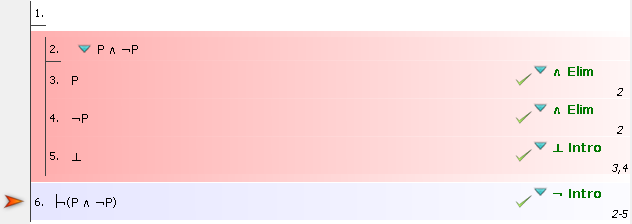
Negation ()

: I am happy

: The sun is shining

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Sentences to cite | Usage |
|  | **Not\_Intro.**  Replaces the current goal by by False and adds the assumption H: | Cite only a single sub proof that begins with the opposite of what you hope to prove and ends with Contra. | Begin a sub proof with the opposite of what you want to prove outside of the sub proof. End the sub proof with Contra.  Cite only the sub proof. |
|  | **Impl\_Elim in H and HO.**  Applies to the two assumptions of the form H: and HO: and adds the new assumption H1: False | Cite only a negation of a negation. | If there is a sentence with at least two negations on it, you can take the negations off, two at a time, with this rule.  Cite only one sentence. |

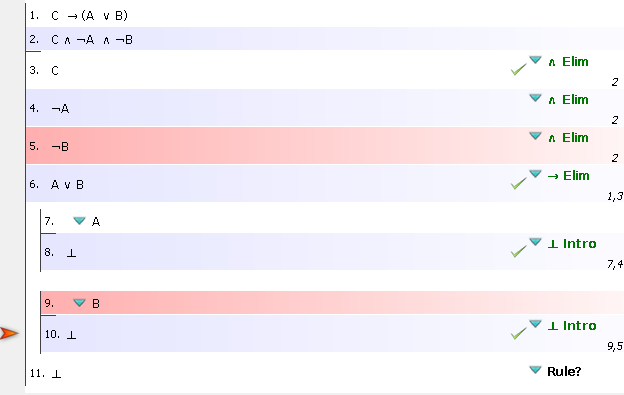
*(¬ Intro) Cite only a single subproof that begins with the opposite of what you hope to prove and ends with Contra*



|  |  |  |  |
| --- | --- | --- | --- |
|  | **PBC**  *Proof by contradiction (Contradiction Introduction)*  Replaces the current goal by False and adds the assumption H: A. | *(⊥ intro)*   * A sentence and, * Exactly that sentence, negated. Cite only two sentences. | Find a sentence and its negation. Cite both and write Contra on a line. |

*(⊥ intro) Cite 4. and 7.*

*(⊥ intro) Cite 5. and 9.*

****

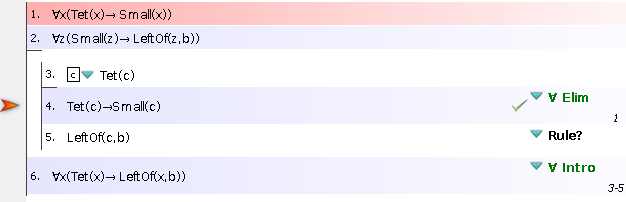
Universal ()

: I am happy

: The sun is shining

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Forall\_Intro.**  Replaces the current goal forall by and adds the variable : to the assumptions. |  |  |
|  | **Forall\_Elim in H with t.**  Applies to an assumption of the form H : forall . It generates a new assumption HO: . |  |  |

*∀ Elim: Cite 1.*



Existential ()

: I am happy

: The sun is shining

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Exists\_Intro with t.**  Replaces the current goal exist by |  |  |
|  | **Exists\_Elim in H.**  Applies to an assumption of the form H : exists . It adss the variable : and the new assumption HO: |  |  |

Proofs practice (ASS1)

**ASS1: Q7**

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*(∨ Elim) Cite 1. And each subproof*

*Each disjunct should be subproof*

*Each subproof should have the same goal as the other (in this case,)*

*The goals in the subproofs should match a new goal outside (in this case, our intermediate conclusion, )*

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*Elim: for each conjunct, cite the assumption.*

*Enter every conjunct from the assumption. On a new line*

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*Intro: Cite as many sentences as you like*

*All must be conjuncts*

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*(∨ Intro). Cite only one prior line, it will be a disjunct.*

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**ASS1: Q8**

A close up of an object

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*-> Intro:*

*Add a new subproof*

*The assumption of the subproof should be the antecedent ()*

*The conclusion of the subproof should be the consequent ()*

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Use . You already have in subproof. Try get into subproof

*-> Elim: Cite a conditional (1.)*

*and a sentence that’s an antecedent of the conditional (4.)*

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*Intro: Cite as many sentences as you like*

*All must be conjuncts*

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**ASS1: Q9**

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*-> Intro:*

*Add a new subproof*

*The assumption of the subproof should be the antecedent ()*

*The conclusion of the subproof should be the consequent ()*

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*(∨ Elim) Cite 1. And each subproof*

*Each disjunct should be subproof*

*Each subproof should have the same goal as the other (in this case)*

*The goals in the subproofs should match a new goal outside (in this case, our intermediate conclusion, )*

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**Lesson 3**

Parse/Syntax Trees

consists of three classes of symbols:

Proposition symbols:

Connective symbols:

Quantifiers:

Punctuation symbols:

Example:

*and bind tighter than*

*The main connective is*

*The main connective of the LHS is*

Parse tree to formula:

Example

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Description automatically generated

*and bind tighter than*

*Main connectives are and*

Example: ASS2 Q4

*A bound variable is one where going up the tree shows a link to a quantifier*

bound

free

bound

free

bound

bound

free

Example: Q2 JUNE 2020

free

free

bound

bound

free

bound

Example: Q2 JUNE 2020

free

bound

free

free

bound

free

bound

**Lesson 4**

NNF, DNF & CNF

DNF: Disjunctive Normal Form.

*Each disjunct is a conjunction*

*Each conjunction is a literal*

*Disjunction () is highest on parse tree*

CNF: Conjunctive Normal Form.

*Each disjunct is a conjunction*

*Each conjunction is a literal*

*Conjunction () is highest on parse tree*

**1. Eliminate arrows**

Material implication:

* (P Q) Q

**2. Deal with negations**

NNF: Negation Normal Form

Double Negation:

P ≡ ¬¬P

¬¬P ≡ P

De Morgan’s Law: Negate all the names **AND** connectives

¬ (P ∨ Q) ≡ ¬P ∧ ¬Q

¬ (P ∧ Q) ≡ ¬P ∨ ¬Q

**3. Deal with conjunction & disjunction**

Distributivity:

* P (Q R) (P Q) (P R)
* P (Q R) (P Q) (P R)

(P Q) (P R)

(P Q) (P R)

**Lesson 5**

Evaluating Sequents

<https://mrieppel.net/prog/truthtable.html>

Example: JUNE 2017 Q1.3

Show that the following sequent is valid by giving an appropriate valuation

**Truth table generator**

(~ p v ~ q)v r, q v r , ~r

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| 1 | 2 | 3 | 4 | 5 | 6 |
|  |  |  |  |  |  |
| T | T | T | T | T | F |
| T | T | F | F | T | T |
| T | F | T | T | T | F |
| T | F | F | T | F | T |
| F | T | T | T | T | F |
| F | T | F | T | T | T |
| F | F | T | T | T | F |
| F | F | F | T | F | T |

*To show that a sequent is not valid, we must find a valuation in which the formulas on the left-hand side are true, but in which the formula on the right-hand side is false.*

**Lesson 6**

Consistency

A set of claims is consistent if it’s logically possible all of them to be true at the same time (not a contradiction)

Let be a theory

is inconsistent if

is consistent if

Example:

All humans are mortal

Some humans are not mortal

*Inconsistent (A)*

All humans are mortal

Simon is immortal

*Consistent, assuming Simon in not human*

All humans are mortal

Simon is immortal

Simon is human

*Inconsistent (A)*

*If Simon is immortal, Simon cannot be human*

**Proof-based:** verify a system using a proof to prove

**Model-based:** computing whether a model satisfies a formula

**Lesson 7**

Satisfaction Relation

<https://en.wikibooks.org/wiki/Formal_Logic/Predicate_Logic/Satisfaction>

We have rules for assigning truth to sentences.

*Every object is true ()*

However, this raises two problems:

[1]

will normally have free variables.

It will normally have free α {\displaystyle \alpha \,\!}

*These are not sentences and do not have a truth value*

[2]

we do not yet have a precise way of saying that φ {\displaystyle \varphi \,\!} is true of every object in the domain

We can solve these problems:

We will need an assignment of objects from the domain to the variables.

We will need to say that a model satisfies (or does not satisfy) a formula with a variable assignment.

|  |  |  |
| --- | --- | --- |
| **Symbol** | **Name** | **Description** |
|  | Phi | Premise/formula |
|  | Psi | Conclusion |
|  | True | Satisfies (models) |
|  | Model M |  |
|  | Environment |  |
|  | Satisfaction relation |  |
|  | Semantic entailment | Is valid if a proof can be found |

The symbol is overloaded

denotes satisfiability

denotes semantic entailment

Whenever ,

Then for all models and look-up tables

Satisfiability:

This is the relation between a wff and a free variable.

We can then define truth in a model in terms of the satisfaction relation:

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Summary :

<https://math.stackexchange.com/questions/2266491/relation-between-satisfaction-relation-and-truth>

**[1]** We have an alphabet and a language, so there is a universe where all the possible combinations lie, empty word and so on.  
**[2]** We define what a formula is, this makes only some words, let's say blue. Now we have some blue things and some colorless.  
**[3]** We choose some blue's as axioms and this is also suddenly make them green.  
**[4]** Then we define a sequent calculus and this makes everything which are reachable from green words suddenly red. Axioms are also now red.

…

Quantifier sentences and the domain:

<https://www.youtube.com/watch?v=fmf8DjhhLrM>

A sentence of the form is true iff at least one object satisfies

A sentence of the form is true iff every object satisfies

Example:

Only satisfied where is a tetrahedron

Example:

Only satisfied where is a small cube

Example:

*Substitute free occurences of with*

This now is a sentence. If the sentence is true, satisfies

Example: ASS2 Q5

Question 5

*All S’s are Q’s.*

*Some Q’s are not S’s*

The model :

*is the set of integers between 0 and 10*

*is the a subset of*

*is the set of even integers between 0 and 10*

Example: ASS2 Q6

*For this question, make a set large enough that can be true*

The model where the sentence is true:

*is the set of all integers*

: The predicate where is less than

The model where the sentence is false:

*is the set of all integers greater than 0 and less than 10*

: The predicate where is less than

Example: ASS2 Q7

*The model :*

*For the satisfaction relation to be satisfied, we need to test that is true for every object in the model .*

*The ordered pairs and*

*The ordered pairs and*

*The ordered pairs and*

*The ordered pairs and*

*The ordered pairs and*

Therefore, the model where satisfies the sentence

**Lesson 8**

Satisfiability

falsifies

satisfies

falsifies

satisfies

**Boolean SAT-isfiability**

<https://www.youtube.com/watch?v=SAXGKCnOuP8>

The 16 cells of the grid are the 16 interpretaions of a, b, c and d

The colors show on the matrix where the current formula is falsified

Consistent CNF Formula

*All blocks are covered*

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **CD** | | | | |
| AB |  | 00 | 01 | 11 | 10 |
| 00 |  |  |  |  |
| 01 |  |  |  |  |
| 11 |  |  |  |  |
| 10 |  |  |  |  |

Inconsistent CNF Formula

*All blocks are covered*

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **CD** | | | | |
| AB |  | 00 | 01 | 11 | 10 |
| 00 |  |  |  |  |
| 01 |  |  |  |  |
| 11 |  |  |  |  |
| 10 |  |  |  |  |

**HORN SAT-isfiability**

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NOT SATISFIABLE

[1] Mark every occurrence of

[2] ARROWS:

WHILE left is marked, mark right

ELSE mark left

[3] IF is marked, NOT SATISFIABLE

ELSE SATISFIABLE

**Lesson 9**

Modal Logic

<https://www.youtube.com/watch?v=EaCLZ9OZzAg>

Extension of propositional Logic that deals with possibility and necessity

Two new operators to WFF:

**Necessity**

*It must be true that p*

|  |  |
| --- | --- |
| John must be lying |  |
| John doesn’t have to be lying |  |
| John must not be lying |  |

**Possibility**

*It may be true that p*

|  |  |
| --- | --- |
| John may be lying |  |
| John can’t be lying |  |
| John may not be lying |  |

s

**Impossibility**

*It can’t be true that p*

Something is contingent if it’s possible for it to be true and false (either necessary or impossible)

Example: elephants are purple

*Contingently false*

**Contingency**

*It may be true that p*

*It can’t be true that p*

Must be necessarily true or necessarily false

Example: Water is

*Necessarily true*

**Analyticity**

*It must be true that p*

Axioms:

Text

Description automatically generated

How each of the above are related:

**Necessity**

**Possibility**

**Impossibility**

**Analyticity**

**requires**

**requires**

**requires**

**requires**

**contradiction**

**Contingency**

Possible worlds

|  |  |  |
| --- | --- | --- |
| John must be lying  *‘in all possible worlds, p is the case’* | **Necessity** | **Possible Worlds** |
| John doesn’t have to be lying |  |  |
| John must not be lying |  |  |

|  |  |  |
| --- | --- | --- |
| John may be lying  *‘in some possible world, p is the case’* | **Possibility** | **Possible Worlds** |
| John can’t be lying  *‘there are no possible worlds, at which p is the case’* |  |  |
| John may not be lying |  |  |

**Lesson 10**

System K

<https://www.youtube.com/watch?v=VLgflakE0lY>

This uses the rules we have learnt so far:

Well-Formed Formulae: If is a wff, is a wff

Boolean Connectives: Binary operators:

Modal Logic: If is a wff, and is a wff

Interpretation: Propositional Logic

The function assigns a truth value to each propositional variable

the sky is blue

pigs can fly

Interpretation: Modal Logic

|  |  |
| --- | --- |
|  | Satisfaction relation |

Possible Worlds

We use a Model

the set of possible worlds

the accessibility relation

*is accessible from*

the assignment function

*P is false in*

*P is false in*

Accessibility Relations

**Necessity**

iff for every such that *,*

*For every world that can access, p is true*

can’t access

world

**Possiblity**

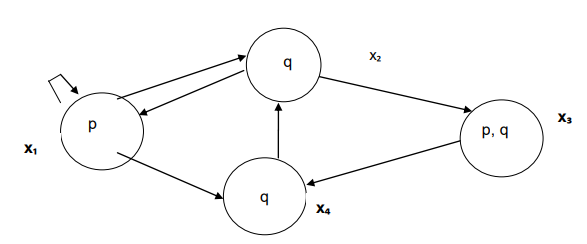
iff there is some world such that *,*

*For every world that can access, p only need be true in ONE world*

need not be true in

Example: ASS3 SEM1 2017

Which world is the formula true?



World

For to be true in , the following must be true:

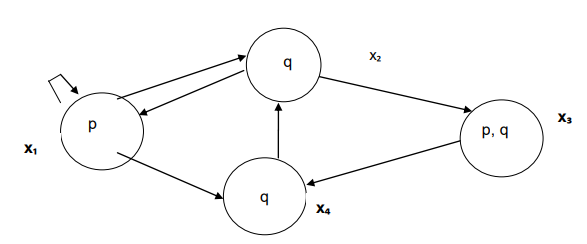
**:**

this holds true as there is at least one world accessible from (mainly ) where is true

**:**

this holds true as every world accessible from (mainly and ) where is true

Example: Q2, Q3 ASS3 SEM1 2017



Which of the following formulas does hold in the Kripke model above?

*There must be at least one world accessible from where is true*

*All worlds accessible from where is true*

*There must be at least one world accessible from where OR is true*

*All worlds accessible from where is true*

*All worlds accessible from where is true*

*There must be at least one world accessible from where is true*

*All worlds accessible from where is true*

*All worlds accessible from where is true*

*All worlds accessible from where AND is true*

Answer: and

Example: Q4 ASS3 SEM1 2017

Which of the following formulas is true in the Kripke model above?

*Does not hold in*

*Does not hold in*

*Holds in all worlds and*

*Does not hold in*

Example: Q5 ASS3 SEM1 2017

Which of the following formulas is false in the Kripke model above?

*Holds in all worlds and*

*Does not hold true in any world, as there is no world where is true*

*Holds in all worlds and*

*Holds in all worlds and*

Answer:

Example: JUNE 2017 Q3.1 (iv)

Diagram

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Determine whether each of the following relations holds in the above Kripke model and give reasons for your answer:

*All worlds accessible from where is true*

*All worlds accessible from where AND is true*

*All worlds accessible from where is true*

*There must be at least one world accessible from where is true*

*No worlds are accessible from therefore the formula holds*